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Restricted Lorentz transformation as an implication of the principle of equivalence

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In this paper, it is shown that the Lorentz transformation is almost deducible from the principle of equivalence The basic postulates, mathematical and physical, are formulated clearly

Introduction

The present paper is concerned with a study of the postulational foundation of the the special theory of relativity. A study of the postulational foundation of any mathematical science is useful for a clearer, better and deeper understanding of the basic postulates of the subject and so helpful for its elegant systematic development. The postulational foundation of a mathematical theory may be studied from different standpoints, *viz.*, philosophical, epistemological, mathematical and physical standpoints. From mathematical and physical standpoints the study of the full implication of a basic postulate and its independence from the others is considered to be very important. A study of the postulational foundation of the special theory of relativity and is not yet a closed chapter A number of papers concerned with its foundation were published (see Whittakers 1953) in the past and are appearing even now (see Bosch 1971).

It is well-known that common relativistic physics may be looked upon as the study of the physical laws (equations) covariants and properties invariant under the group of Lorentz tranformations, which are generally deduced from two basic physical postulates, *viz* ,

(P 1) the velocity of light in vaccuum is independent of the velocity of the source of light and the direction of propagation, *etc.* (the principle of the constancy of velocity of light)

(P II) "there exists a triply infinite set of reference systems moving rectilinearly and uniformly relative to one another in which the phenomena occur in an identical manner". (the principle of equivalence or relativity ; Pauli, 1963).

Of course, in usual deductions of Lorentz transformations, besides these two basic physical postulates, a mathematical postulate, *viz*,

(M.1) all transformations are linear, (the principle of linearity)

is always introduced.

In most of the developments of the special theory of relativity, the linearity of coordinate-transformations in the space-time continuum is generally introduced as a simplifying mathematical assumption In the first paper of Einstein (1905) on the subject, the linearity of the transformations is attributed to homogeneity of space and time. Later (Einstein 1955) it is considered as a consequence of the principle of constancy of velocity of light According to Pauli (1963), it is due to two simple physical facts namely, a uniform rectilinear motion in one inertial frame is rectilinear in any other inertial frame and furthermore 'finite coordinates remain finite in transformations' Several attempts to deduce linearity from these two simple postulates or from other simple postulates are due to Weyl (see Fock 1959) and others (see Mukherjee 1936) Recently, it has been shown (Dutta, 1970) that linearity of transformations is implied by the simple fact of finite coordinates remaining finite under transformations, which is a sufficient condition for the principle of equivalence (relativity) and almost a necessary consequence of this principle The fact that finite coordinates remain finite may be referred to as a principle of finiteness of coordinates Naturally, after the recent investigation (Dutta, 1970), the question, what more information about the form of trnsformation rules is implied by the principle of equivalence alone becoming important In this paper it is shown that the usual form of restricted Lorentz tranformations are deducible from the principle of equivalence alone

Of course, several attempts were already made by a number of investigators like Ignatowsky, Frank & Rothe (see Pauli 1963) and Sen (1936) to deduce restricted Lorentz transformations from the principle of equivalence without the supplementary postulate of the constancy of the velocity of light.

The investigations of Ignatowsky, Frank & Rothe are from general group-theoretic considerations based on the following postulates

(P_a) the transformations must form a one-parametric homogeneous linear group,

(P_b) the velocity of a reference system K relative to K' is equal and opposite to that of K' relative to K,

(P_c) the contraction of length of a rod at rest in K' and observed in K is equal to that of the same rod at rest K and observed in K' (see Pauli 1963).

The postulate (P_a) is of mathemmatical nature and commonly assumed for coordinate transformation, considered in wellknown geometries It contains the principle (M I) The postulate (P_b) states a fact of experience, very simple but not implied by the other two postulates directly. The last one P_c may be taken to be a simple consequence of the principle of equivalence as formulated by Pauli (1963), provided the contraction of length is considered as a physical quantity.

In Sen's deduction of restricted Lorentz transformation (1936) the principle of equivalence is supplemented with a general principle of continuity, which is a mathematical principle, common to all branches of classical physics and with the principle of equivalence of velocity measurements in different inertial frames. The restricted Lorentz transformations are also deduced by Sen (1936) from the principle of equivalence and the principle of finiteness for velocities, *i e*, the fact that a finite velocity in an inertial frame remains finite in any other inertial frame. He obtained the expressions for transformations in terms of a maximum velocity, which was identified with the velocity of light in the Michelson-Morley experiment An investigation like that of Sen (1936) is of a great importance in a study of foundation of special theory of relativity as it throws light on the great significance of the principle of equivalence.

The investigations of Sen (1936) is a *step by step* construction of transformation by only elementary mathematics, from the following principles in addition to the principle (MI)

(SI) The length of a rod at rest in K' and observed in K is equal to that of the same rod at rest in K and observed in K'.

(SII) The time-interval in a clock at rest in K' and observed in K is equal to that in the same clock in K and observed in K'

(SIII) There exists a velocity V in K such that when measured by the observer at rest in K' it will be exactly the same as that measured by the observer at rest in K measuring the same velocity V in K'.

(SIV) The physical quantities like coordinates, time *etc.* are continuous functions of velocities

The postulate (SI) is the same as (Pc) The postulate (SII) may also be considered as a simple direct consequence of the principle of equivalence. The postulate (SIII) is considered by Sen as an extension of the principle of equivalance The postulate (SIV) is the principle of continuity, a postulate of mathematical nature common to all branches of classical physics. Of course, Sen also assumed implicitly that transformations are always non-singular

In the same paper (1936) Sen also deduced the restricted Lorentz transformations by replacing the postulates (SIII) and (SIV) by other two principles *viz.*,

(SIII)′ Transformations from a groupoid

(SIV)′ Every velocity finite with respect to K' must also be finite with respect to K

or (SIV)′ The addition of two positive velocities are always positive.

Now the postulate (SIII)′ along with implicit assumption of non-singularity of transformations leads to the postulate that the transformations form a loop. As the property of associativity is inherent for the class of linear transformations so, these form a group The postulate (SIV) is considered as an extension of the principle of finiteness. The postulate (SIV)′ is to be taken as a simple fact of experience.

In the present paper, it is shown that the principle of equivalence yields the restricted Lorentz transformations and the existence of a preferred velocity, which may be interpreted as the maximum of common physical velocities without the supplement of any other mathematical or physical postulate, if along with (SIII)′ of Sen (1936) it is only assumed that the transformation must form a loop of linear transformations The equivalence of measurements of lengths and of time-intervals being the same in every pair of inertial frames, is implied by the principle of equivalence (SII) Here, the mathematical calculations are very simple, brief and elementary Along with Sen (1936), the mathematical axiom about the set of teransformations is that it is a loop. Of course, the set of transformations obtained may be shown to be a group, as the associativity is obvious but not necessary for calculations

Basic Postulates

In a discussion of the postulational foundation of a subject, it is always essential to formulate clearly and explicitly the basic postulates used. In critical discussions of mathematical theories of a physical science the postulates may be divided into two classes, *viz.*, the mathematical postulates and the physical postulates For elegance of the theory, any one of the postulates is independent of the other in the sense that none is implied by others But, postulates must be compatible with one another. For this, the mathematical postulates should be so formulated that the basic ideas of physical postulates are expressible by mathematical equations or more precisely by mathematical relations and thus mathematical theories may be built up.

With the above points in view, the basic mathematical postulates are :

(MP I) The space-time continuum is isomorphic to an arithmetic space of dimension four, R_4.

(MP II) Transformations are automorphisms of R_4 forming an one-parametric homogeneous linear groupoid and keeping points of a two-dimensional linear subspace of R_4 fixed

(MP III) The parameters form a set of $\boldsymbol{V}$ in a real (open) line R.

At the begining, it is to be mentioned that in the present discussion only algebraic structure of R_4 and R are really relevant. As the set $\boldsymbol{V}$ is contained in R, ordinary rules of addition, multiplication, etc. are implied amongst the elements of $\boldsymbol{V}$ but the characteristic rules of algebraic operation will be obtained in subsequent discussion. Since transformations are automorphisms, inverse of every transformation and the identity transformation exist Thus, actually transformations form a 'loop'. Associativity of the product of transformations is not at all necessary for the present discussion but is easily deducible from matrix representation and so is not postulated Here it is to be noted that these mathematical postulates are assumed at least explicitly in all the earlier works mentioned in the previous section

The main calculation is based on two physical postulates, identical with (SI) and (SII) to be referred to hereafter as (PPI) and (PPII) In a subsequent section the postulate (PPII) is replaced by (P_b), to be referred to as (PPII)$'$, and its implications are studied critically.

Main Calculations

In the space-time continuum, coordinate systems and so the frame of references are to be introduced in the usual fashion (Veblen & Whitehead 1953). Now, by proper choice of frames, the linear subspace of fixed points is to be taken as the $y-z$ plane Then according to (MPI) we are to consider the loop of coordinate transformations (automorphisms) of R_4 given by :

$$\begin{aligned} x' &= \alpha x+\beta t, \; y' = y \quad z' = z, \\ t' &= \gamma x + \delta t. \end{aligned} \qquad \ldots \quad (1)$$

where x', y', z', t' are the coordinates in a frame K' of point of which the coordinates in the frame K are x, y, z, t; the frame K' is moving with constant velocity v, relative to K in the positive direction of x-axis and at $t = 0$ the origin and axes are coincident and the coefficient α, β, γ, δ are depending on a parameter by (MPII) As the γ-coordinate of the origin of K' at time t is vt so we have

$$\beta = -\alpha v \qquad \ldots \quad (2)$$

Thus, without loss of any generality, the parameter may be identified with v and so α, β γ, δ are functions of v

Let at time t_0 in K a rod be kept at rest along the x-axis, and the length be measured from K'. The contraction of length is then given by the factor α.

Now, the inverse transformation exists by (MP II), and is given by

$$\begin{aligned} x &= \frac{\delta}{\Delta} x' - \frac{\beta}{\Delta} t', \quad y = y', \quad z = z' \qquad (3) \\ t &= -\frac{\gamma}{\Delta} x' + \frac{\alpha}{\Delta} t', \\ \Delta &= \alpha\beta - \gamma\delta. \end{aligned}$$

So, the contraction is given by δ/Δ.

As a consequence of (PP I), we have

$$\alpha = \frac{\delta}{\Delta}. \tag{4}$$

Similarly from calculations of dilatations of time-interval, we get

$$\delta = \frac{\alpha}{\Delta}. \tag{5}$$

By (4) and (5), we get

$$\alpha\delta = \frac{\alpha\delta}{\Delta^2} \text{ or } \Delta^2 = 1, \text{ ie } \Delta = \pm 1 = \xi, \tag{6}$$

$$\alpha = \pm\delta = \xi\delta, \tag{7}$$

where ξ is taken to have a value $+1$ and -1 and the correct value will be determined afterwards

By (MP II), the set of transformations contain identity transformation given by $v = 0$, so Δ or ξ must have the value 1 in same case. Now, it is to be noted that the postulate of continuity at once leads to the value, $\Delta = 1$ always The notion of continuity necessitates the introduction of postulates for topological structure for R_4 and R. In the present discussion, no such postulate has been introduced So, we have to show that Δ or ξ is not equal to -1

Then, the equation (1) can be written as

$$\begin{aligned} x' &= \alpha(x-vt), \quad y' = y, \quad z' = z, \\ t' &= \alpha(\xi t + \epsilon x), \end{aligned} \qquad \dots \quad (8)$$

where
$$\epsilon = \frac{\gamma}{\alpha} \qquad \dots \quad (9)$$

Let us consider another frame K'' moving with a constant velocity v' relative to K' in the direction of x'-axis and at $t' = 0$, K'' is coincident with K'. Then, if x'', y'', z'', t'' are coordinates in K'', we have

$$x'' = \alpha'(x'-v't') = \alpha\alpha'(1-cv')\left(x - \frac{v+v'\xi}{1-\epsilon v'}\, t,\right)$$

$$y'' = y' = y, \quad z'' = z' = z$$

$$t'' = \alpha'(v'\xi + \epsilon' x') = \alpha\alpha'(1-c'v)\left(t - \frac{\xi+\epsilon'}{1-\epsilon' v}\, x\right) \qquad \dots \quad (10)$$

By (MP II), a transformation of coordinates from K to K'' for a value, v'' (say), to be interpreted as the velocity of K'' relative to K, exists and is given by the equation (1) as

$$x'' = \alpha''(x-v''t), \quad y'' = y, \quad z'' = z,$$

$$t'' = \alpha''(t-\epsilon''x), \qquad \dots \quad (11)$$

By comparing (10) with (11), we have

$$1-\epsilon v' = 1-\epsilon' v \qquad (12)$$

$$v'' = \frac{v+v'\xi}{1-\epsilon v'} \qquad (13)$$

$$\epsilon'' = \frac{\xi\epsilon+\epsilon'}{1-\epsilon' v} \qquad (14)$$

$$\alpha'' = \alpha\alpha'(1-\epsilon v') = \alpha\alpha'(1-\epsilon' v) \qquad (15)$$

By (12), we have

$$\frac{v}{\epsilon} = \frac{v'}{\epsilon'} \qquad (16)$$

Now, v/ϵ is a function of v only and v'/ϵ' that of v' only and so each ratio is independent of v or v'. As ϵ and ϵ' have the dimension of the reciprocal of velocity, so,

$$\frac{v}{\epsilon} = \frac{v'}{\epsilon'} = \frac{V^2}{\eta} \qquad \dots \quad (17)$$

where V is a velocity, same for all frames and independent of the velocity v or v' of the frame and η is a pure member. Now, the magnitude of η may be observed in V^2 and stands for $+1$ or -1.

Then, by (13), (14) and (17) we have,

$$v'' = \frac{v+v'\xi}{1-\eta\dfrac{vv'}{V^2}} \qquad \dots \quad (18)$$

$$\epsilon'' = \frac{\dfrac{\eta}{V}(v\xi+v')}{1-\eta\dfrac{vv'}{V^2}} \qquad \dots \quad (19)$$

By (18) and (19), we have

$$\frac{V^2}{\eta} = \frac{v''}{\epsilon''} = \frac{V^2}{\eta}\,\frac{v+v'\xi}{v\xi+v'}, \quad \text{or} \quad \xi = 1 \qquad \dots \quad (20)$$

Then by (2), (3), (6), and (20), we have

velocity of K relative to $K' = -v$ (21)

This is the postulate (PP II)$'$ or (Pb).

Thus, we have the velocity addition theorem as

$$v'' = \frac{v+v'}{1-\eta \frac{vv'}{V^2}}, \tag{22}$$

By (9) and (17),

$$\gamma = \eta \frac{v}{V^2} \alpha \tag{23}$$

Then, by (2), (3), (6), (20) and (23)

$$1 = \Delta = \alpha^2 \left(1+\eta \frac{v^2}{V^2}\right)$$

or

$$\alpha = \frac{1}{\sqrt{1+\eta \frac{v^2}{V^2}}} \tag{24}$$

as x-axis and x'-axis have the same orientation.

If possible, let $\eta = 1$

By (24), v may have any real value, *i e.* $\boldsymbol{V}$ coincides with R

Then, by (23), v'' does not exist when $v' = -V/v$ It contradicts (MP II). So, Δ or ξ should not be taken as 1.

When $\eta = -1,$ ... (25)

by (MP I), $v^2 < V^2$, *i.e.*, $\boldsymbol{V} = (-V, V)$ ... (26)

and the transformation (1) takes the explicit form ·

$$x' = \frac{1}{\sqrt{1-\frac{v^2}{V^2}}} (x-vt),$$

$$y' = y, \quad z' = z \qquad \dots \tag{27}$$

$$t = \frac{1}{\sqrt{1-\frac{v^2}{V^2}}}$$

This passes over to the well-known restricted Lorentz transformation if V be identified as the velocity c of light in vacuum, either by the postulate (P II) or by direct appeal to the physical experience like the Michelson-Morley experiment,

By (22),
$$v'' = \frac{v+v'}{1+\frac{vv'}{V^2}} \qquad \dots \quad (28)$$

is the well-known velocity addition rule. Now, as either v or v' or both approach V, then v'' approaches V *i.e.*, V is the least upper bound and also the upper limiting point of the velocities. Similarly V is the greatest lower bound and the lower limiting point of the velocities. If for either v or v' or both the value V be substituted, we get $v'' = V$. But, the interpretation of $v = V$ as the velocity of a frame leads to difficulties in transformation formula (27).

Other Calculations

It is easy to see that all the above results are deducible in a slightly simpler way when the postulate (PP II) is replaced by (PP II)′ In this case, as in the above, by PP(I), we have the relation (4). By (PP II)′ and (3), we have

$$-v = \frac{\beta}{\delta} = -\frac{\alpha v}{\delta}, \quad \text{or} \quad \alpha = \delta \qquad \dots \quad (29)$$

Then, by (23) and (29), we have
$$\Delta = 1 \qquad \dots \quad (30)$$

Then, we have (5) directly, *i.e.*, the postulate (PP II) is satisfied. The rest of calculations are similar to those in the preceeding section. Of, course, they are a bit simpler

Conclusions

It is interesting to note that $\boldsymbol{V}$ becomes an Abelian group under modified addition defined by (28) So is a set in R but not even embedded in R as an Abelian group. How the multiplication rule is modified is never discussed in the literature

In the earlier paper it is shown that linearity of transformations are almost a consequence of the principle of the equivalence. The postulates (PP I) and (PP II) are simple consequences of the same principle. Thus, the Lorentz transformation is almost deducible from the said principle.

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